On the dual of complex Ol'shanskiĭ semigroups

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Introduction

Let G be a connected Lie group which sits in its universal complexification $G_{\mathbb{C}}$. If $\mathfrak{g}$ denotes the Lie algebra of G and $W \subseteq \mathfrak{g}$ is a non-empty open $\mathrm{Ad}(G)$-invariant convex cone containing no affine lines, then a *complex Ol'shanskiĭ semigroup* is defined by $S = G\exp(iW) \subseteq G_{\mathbb{C}}$. One knows that S is an open subsemigroup of $G_{\mathbb{C}}$ with holomorphic multiplication, and moreover S is invariant under the antiholomorphic involution $g \mapsto g^* = \overline{g}^{-1}$, where $\overline{g}$ indicates complex conjugation in $G_{\mathbb{C}}$. In particular, $(S,*)$ is an *involutive semigroup.*

If $\mathcal{H}$ is a Hilbert space and $B(\mathcal{H})$ denotes the bounded operators on it, then a *holomorphic representation* $(\pi, \mathcal{H})$ of S is a holomorphic semigroup homomorphism $\pi\colon S \to B(\mathcal{H})$ with total image and which satisfies $\pi(s^*) = \pi(s)^*$ for all $s \in S$. A mapping $\alpha\colon S \to [0,\infty[$ satisfying $\alpha(s^*) = \alpha(s)$ and $\alpha(st) \leq \alpha(s)\alpha(t)$ for all $s,t \in S$ is called an *absolute value* of S. We call a holomorphic representation α-*bounded* for some absolute value α if $\|\pi(s)\| \leq \alpha(s)$ holds for $s \in S$.

The α-bounded holomorphic representations of S can be modelled via a certain C^*-algebra $C_h^*(S,\alpha)$ (cf. Definition I.3, Lemma II.6), i.e., there is a natural correspondence between α-bounded holomorphic representations of S and non-degenerate representations of $C_h^*(S,\alpha)$. An important result of K.-H. Neeb asserts that these C^*-algebras $C_h^*(S,\alpha)$ are CCR (cf. [Ne99, Ch. XI]). If we denote by $\widehat{S}_\alpha$ the set of equivalence classes of irreducible α-bounded representations of S, we therefore obtain a bijection $\widehat{S}_\alpha \cong C_h^*(S,\alpha)\widehat{\ }$. The hull-kernel topology on $C^*(S,\alpha)\widehat{\ }$ thus gives rise to a topology on $\widehat{S}_\alpha$ denoted by $\mathcal{T}_{hk}^\alpha$.

On the other hand irreducible holomorphic representations of S are obtained in a unique fashion by analytic continuation of unitary highest weight representations of G and vice versa. Thus we may think of $\widehat{S}_\alpha$ as a certain subset of $\widehat{G}$ and the topology on $\widehat{G}$ induces a topology on $\widehat{S}_\alpha$ denoted by $\mathcal{T}_G^\alpha$. Finally the parametrization of $\widehat{S}_\alpha$ by a certain subset of highest weights $HW_\alpha \subseteq i\mathfrak{t}^*$, where $\mathfrak{t}$ denotes a compactly embedded Cartan subalgebra, gives a topology on $\mathcal{T}_e^\alpha$ obtained by the euclidean topology on HW_α. Our main result (cf. Theorem II.24) then asserts that one has

$$\mathcal{T}_{hk}^\alpha \subseteq \mathcal{T}_G^\alpha \subseteq \mathcal{T}_e^\alpha.$$

Moreover, the induced Borel structures are all the same and for generic absolute values even all the topologies coincide. These results are obtained by a combination of holomorphic representation theory (cf. [Ne99]), standard structure theory of C^*-algebras (cf. [Dix82]) and real analysis methods (boundary values of Poisson transforms). Our results imply in particular that the CCR-algebra $C_h^*(S,\alpha)$ has separated dual for generic absolute values, which allows us to identify $C_h^*(S,\alpha)$ with the C^*-algebra defined by the continuous field of elementary C^*-algebras $\big(\mathcal{K}(\mathcal{H}_\lambda)\big)_{\lambda\in HW_\alpha}$. Further we explain what our result means for the abstract representation theory of complex Ol'shanskiĭ semigroups.

Finally we give a criterion for an absolute value α in order that the C^*-algebra $C_h^*(S,\alpha)$ has continuous trace (cf. Proposition II.33).

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Involutive semigroups

Definition I.1. (Involutive semigroups) (a) An *involutive semigroup* is a semigroup S equipped with an involutive antiautomorphism $s \mapsto s^*$.

If S does not have an identity, we set $S_{\mathbf{1}} := S \dot\cup \{\mathbf{1}\}$. Then the prescription $s\mathbf{1} = \mathbf{1}s = s$ for all $s \in S$ and $\mathbf{1}^* = \mathbf{1}\mathbf{1} = \mathbf{1}$ together with the structure on S turns $S_{\mathbf{1}}$ into an involutive semigroup.

(b) Let $\mathcal{H}$ be a pre-Hilbert space. A *hermitian representation* $(\pi, \mathcal{H})$ of S is a semigroup homomorphism $\pi: S \to \mathrm{End}(\mathcal{H})$ such that $\langle \pi(s^*).v, w\rangle = \langle v, \pi(s).w\rangle$ holds for all $s \in S$ and $v, w \in \mathcal{H}$. If in addition $\mathcal{H}$ is a Hilbert space and $\pi(S) \subseteq B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the bounded operators on $\mathcal{H}$, then we call $(\pi, \mathcal{H})$ a *representation* of S.

(c) If S is an involutive semigroup, then an *absolute value* on S is a mapping $\alpha: S \to [0, \infty[$ which satisfies

$$\alpha(s) = \alpha(s^*) \qquad \text{and} \qquad \alpha(st) \leq \alpha(s)\alpha(t)$$

for $s, t \in S$. The collection of all absolute values on S is denoted by $\mathcal{A}(S)$.

(d) A representation $(\pi, \mathcal{H})$ of S is called α-*bounded* for some $\alpha \in \mathcal{A}(S)$ if $\|\pi(s)\| \leq \alpha(s)$ holds for all $s \in S$. ∎

Definition I.2. (Positive definite functions) Let S be an involutive semigroup.

(a) A function $\varphi: S \to \mathbb{C}$ is called *positive definite* if for all finite sequences $s_1, \ldots, s_n$ in S and $c_1, \ldots, c_n$ in $\mathbb{C}$ one has

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_k s_j^*) \geq 0.$$

(b) Let φ be a positive definite function on S. For each $t \in S$ we define a function $\varphi_t(s) := \varphi(st)$ on S and set $\mathcal{H}_\varphi^0 = \mathrm{span}\{\varphi_t : t \in S\}$. Then the prescription

$$\langle \sum_{j=1} c_j \varphi_{t_j}, \sum_{k=1} l_k \varphi_{s_k} \rangle = \sum_{j,k} c_j \overline{l_k} \varphi(s_k^* t_j)$$

defines a pre-Hilbert space structure on $\mathcal{H}_\varphi^0$. We denote by $\mathcal{H}_\varphi$ the closure of $\mathcal{H}_\varphi^0$ and note that $\mathcal{H}_\varphi$ also consists of functions on S (cf. [Ne99, Ch. III]).

(c) If φ is positive definite, then the prescription

$$\pi_\varphi^0: S \to \mathrm{End}(\mathcal{H}_\varphi^0), \quad (\pi_\varphi^0(s).f)(t) = f(ts)$$

defines a hermitian representation $(\pi_\varphi^0, \mathcal{H}_\varphi^0)$ of S on the pre-Hilbert space $\mathcal{H}_\varphi^0$. In the case where all operators $\pi_\varphi^0(s)$, $s \in S$, are bounded, the hermitian representation $(\pi_\varphi^0, \mathcal{H}_\varphi^0)$ extends to a representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of S which is also given by right translations (cf. [Ne99, Th. III.1.3]).

(d) If $\alpha \in \mathcal{A}(S)$, then a positive definite function φ on S is called α-*bounded* if

$$(\exists C > 0)(\forall s, t \in S) \qquad |\varphi(t^* s t)| \leq C\alpha(s)\varphi(t^* t).$$

Note that C may be replaced by $C = 1$ and that the α-boundedness of φ is equivalent to the α-boundedness of $(\pi_\varphi, \mathcal{H}_\varphi)$ (cf. [Ne99, Th. III.1.19]).

We denote by $\mathcal{P}(S, \alpha)$ the convex cone of all α-bounded positive definite functions on S. ∎

Definition I.3. (Enveloping C^*-algebras) Let S be an involutive semigroup and α be an absolute value on it. Then for a subset $\mathcal{E} \subseteq \mathcal{P}(S, \alpha)$ we set

$$\mathcal{H}_{\mathcal{E}} := \widehat{\bigoplus}_{\varphi \in \mathcal{E}} \mathcal{H}_\varphi.$$

Then $\pi_{\mathcal{E}}(s) := \oplus_{\varphi \in \mathcal{E}} \pi_\varphi$ defines an α-bounded representation of S on $\mathcal{H}_{\mathcal{E}}$. We denote by $C^*(S, \alpha, \mathcal{E})$ the closure of $\mathrm{span}(\pi_{\mathcal{E}}(S))$ in $B(\mathcal{H}_{\mathcal{E}})$ and note that this is a C^*-subalgebra of $B(\mathcal{H}_{\mathcal{E}})$. Note that the mapping

$$j: S \to C^*(S, \alpha, \mathcal{E}), \quad s \mapsto \pi_{\mathcal{E}}(s)$$

has total image by definition. If $\mathcal{E} = \mathcal{P}(S, \alpha)$, then we set $C^*(S, \alpha) := C^*(S, \alpha, \mathcal{E})$. ∎

Lemma I.4. *Every representation $(\widetilde{\pi}, \mathcal{H})$ of $C^*(S,\alpha)$ gives via*

$$\widetilde{\pi} \mapsto \pi := \widetilde{\pi} \circ j$$

rise to a α-bounded representation $(\pi, \mathcal{H})$ of S. Moreover, this correspondence is bijective.

Proof. [Ne99, Th. III.2.9]. ■

Representations of involutive semigroups can be described in terms of positive definite functions, those of C^*-algebras by positive functionals. We describe now a correspondance between these two pictures.

For every C^*-algebra $\mathcal{A}$ we denote by $\mathcal{A}_{\mathbf{1}}$ its unification (adjoining of a unit element if there was none). Note that every positive functional f on $\mathcal{A}$ extends uniquely to a positive functional $\widetilde{f}$ on $\mathcal{A}_{\mathbf{1}}$ with $\widetilde{f}(\mathbf{1}) = \|f\|$ (cf. [Dix82, Sect. 2.4.3]).

We set

$$E(C^*(S,\alpha)) := \{f \in C^*(S,\alpha)' : f \text{ positive functional; } \|f\| = 1\},$$

and

$$\mathcal{P}(S,\alpha)_{\mathbf{1}} := \{\varphi|_S : \varphi \in \mathcal{P}(S_{\mathbf{1}},\alpha),\ \varphi(\mathbf{1}) = 1\}.$$

Note that for all $\varphi \in \mathcal{P}(S,\alpha)_{\mathbf{1}}$ one has $\varphi \in \mathcal{H}_\varphi$ (cf. [Ne99, Prop. III.1.23]).

Theorem I.5. *If we equip $\mathcal{P}(S,\alpha)_{\mathbf{1}}$ with the topology of pointwise convergence on S and $E(C^*(S,\alpha))$ with the weak-$*$-topology with respect to $C^*(S,\alpha)$, then the mapping*

$$\Psi: \mathcal{P}(S,\alpha)_{\mathbf{1}} \to E(C^*(S,\alpha)), \quad \varphi \mapsto f_\varphi;\ f_\varphi(x) := \langle \widetilde{\pi_\varphi}(x).\varphi, \varphi\rangle$$

is a homeomorphism.

Proof. First we show that Ψ is defined. Since f_φ is obviously positive, we have to show that $\|f_\varphi\| = 1$ or equivalently $\widetilde{f_\varphi}(\mathbf{1}) = 1$. But since $(\widetilde{\pi}_\varphi, \mathcal{H}_\varphi)$ is non-degenerate (cf. Lemma I.4), it extends naturally to a representation of $C^*(S,\alpha)_{\mathbf{1}}$ by setting $\widetilde{\pi_\varphi}(\mathbf{1}) = \mathrm{id}$. Thus

$$\widetilde{f_\varphi}(\mathbf{1}) = \langle \widetilde{\pi_\varphi}(\mathbf{1}).\varphi, \varphi\rangle = \langle \varphi, \varphi\rangle = \varphi(\mathbf{1}) = 1,$$

and so Ψ is defined.

We describe now the construction of Ψ^{-1}. Let $f \in E(C^*(S,\alpha))$. We consider $C^*(S,\alpha)$ as an involutive semigroup and f as a positive definite function on it. In this sense let $(\pi_f, \mathcal{H}_f)$ be the representation of Definition I.2(c). Since f extends to a positive definite function $\widetilde{f}$ on $C^*(S,\alpha)_{\mathbf{1}}$, it follows from [Ne99, Prop. III.1.23] that $f \in \mathcal{H}_f$ and that π_f extends to a representation of $C^*(S,\alpha)_{\mathbf{1}}$ denoted by the same symbol. Note that $f(x) = \langle \pi_f(x).f, f\rangle$ for all $x \in C^*(S,\alpha)_{\mathbf{1}}$. Now we define $\varphi_f \in \mathcal{P}(S,\alpha)$ by $\varphi_f(s) := \langle \pi_f(j(s)).f, f\rangle$ for all $s \in S$. From the fact that f extends to a positive definite function on $C^*(S,\alpha)_{\mathbf{1}}$, it follows that φ_f extends to a positive definite function on $S_{\mathbf{1}}$ by setting $\varphi_f(\mathbf{1}) = f(\mathbf{1}) = 1$.

For the bijectivity of Ψ we now have to show that

$$(\forall \varphi \in \mathcal{P}(S,\alpha)_{\mathbf{1}}) \quad \varphi = \varphi_{f_\varphi} \tag{1.1}$$

$$(\forall f \in E(C^*(S,\alpha)) \quad f = f_{\varphi_f}. \tag{1.2}$$

By the definition of f_φ we have

$$f_\varphi(j(s)) = \langle \widetilde{\pi_\varphi}(j(s)).\varphi, \varphi\rangle = \varphi(s) \tag{1.3}$$

for all $s \in S$. Thus by the definition of φ_f we get that

$$\varphi_{f_\varphi}(s) = f_\varphi(j(s)) = \varphi(s)$$

for $s \in S$, proving (1.1). By the same reasoning we obtain that

$$f_{\varphi_f}(j(s)) = \varphi_f(s) = f(j(s))$$

so that f and f_{φ_f} coincide on the total set $j(S)$. Since both are continuous, they coincide, proving (1.2).
Continuity of Ψ: If $\varphi_n \to \varphi$ pointwise on S, then (1.3) implies that $f_{\varphi_n} \to f_\varphi$ on the dense set $\mathrm{span}\{j(S)\}$. Since $E(C^*(S,\alpha))$ is a bounded subset of continuous functions on $C^*(S,\alpha)$ it thus follows that $f_{\varphi_n} \to f_\varphi$ in the weak-$*$-topology of $C^*(S,\alpha)$.
Continuity of Ψ^{-1}: If $f_{\varphi_n} \to f_\varphi$ pointwise on $C^*(S,\alpha)$, we conclude from (1.3) that $\varphi_n \to \varphi$ pointwise on S. ■

II. The dual of complex Ol'shanskiĭ semigroups

In this section we focus our interest on a concrete class of involutive semigroups, namely complex Ol'shanskiĭ semigroups, which may be thought as generalizations of complex Lie subsemigroups of complex Lie groups.

Complex Ol'shanskiĭ semigroups

Definition II.1. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra.
(a) An element $X \in \mathfrak{g}$ is called *elliptic* if $\operatorname{ad} X$ is semsimple with purely imaginary spectrum. Accordingly we call a subset $W \subseteq \mathfrak{g}$ *elliptic* if all its elements are elliptic.
(b) A subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is said to be *compactly embedded*, if $\langle e^{\operatorname{ad} \mathfrak{a}} \rangle$ is relatively compact in $\operatorname{Aut}(\mathfrak{g})$. Note that a subalgebra is compactly embedded if and only if it is elliptic. ■

Definition II.2. Let $\emptyset \neq W \subseteq \mathfrak{g}$ be an open convex $\operatorname{Inn}(\mathfrak{g})$-invariant elliptic cone and $\overline{W}$ its closure. Let $\widetilde{G}$, resp. $\widetilde{G}_{\mathbb{C}}$, be the simply connected Lie groups associated to $\mathfrak{g}$, resp. $\mathfrak{g}_{\mathbb{C}}$, and set $G_1 := \langle \exp \mathfrak{g} \rangle \subseteq \widetilde{G}_{\mathbb{C}}$. Then Lawson's Theorem (cf. [HiNe93, Th. 7.34, 35]) says that the subset $\Gamma_{G_1}(\overline{W}) := G_1 \exp(i\overline{W})$ is a closed subsemigroup of $G_{\mathbb{C}}$ and the polar map

$$G_1 \times \overline{W} \to \Gamma_{G_1}(\overline{W}), \quad (g, X) \mapsto g \exp(iX)$$

is a homeomorphism.

Now the universal covering semigroup $\Gamma_{\widetilde{G}}(\overline{W}) := \widetilde{\Gamma}_{G_1}(\overline{W})$ has a similar structure. We can lift the exponential function $\exp: \mathfrak{g} + i\overline{W} \to \Gamma_{G_1}(\overline{W})$ to an exponential mapping $\operatorname{Exp}: \mathfrak{g} + i\overline{W} \to \Gamma_{\widetilde{G}}(\overline{W})$ with $\operatorname{Exp}(0) = \mathbf{1}$ and thus obtain a polar map $\widetilde{G} \times \overline{W} \to \Gamma_{\widetilde{G}}(\overline{W})$, $(g, X) \mapsto g \operatorname{Exp}(iX)$ which is a homeomorphism.

If G is a connected Lie group associated to $\mathfrak{g}$, then $\pi_1(G)$ is a discrete central subgroup of $\Gamma_{\widetilde{G}}(\overline{W})$ and we obtain a covering homomorphism $\Gamma_{\widetilde{G}}(\overline{W}) \to \Gamma_G(\overline{W}) := \Gamma_{\widetilde{G}}(\overline{W})/\pi_1(G)$ (cf. [HiNe93, Ch. 3]). It is easy to see that there is also a polar map $G \times \overline{W} \to \Gamma_G(\overline{W}), (g, X) \mapsto g \operatorname{Exp}(iX)$ which is a homeomorphism. The semigroups of the type $\Gamma_G(\overline{W})$ are called *complex Ol'shanskiĭ semigroups*.

The subset $\Gamma_G(W) \subseteq \Gamma_G(\overline{W})$ is an open semigroup carrying a complex manifold structure such that the multiplication is holomorphic. Moreover there is an involution on $\Gamma_G(\overline{W})$ given by

$$^*: \Gamma_G(\overline{W}) \to \Gamma_G(\overline{W}), s = g \operatorname{Exp}(iX) \mapsto s^* = \operatorname{Exp}(iX) g^{-1}$$

being antiholomorphic on $\Gamma_G(W)$ (cf. [HiNe93, Th. 9.15] for a proof of all that). Thus both $\Gamma_G(W)$ and $\Gamma_G(\overline{W})$ are involutive semigroups. ■

From now on we denote by S an open complex Ol'shanskiĭ semigroup $\Gamma_G(W)$ and by $\overline{S}$ its "closure" $\Gamma_G(\overline{W})$.

Definition II.3. A *holomorphic representation* $(\pi, \mathcal{H})$ of a complex Ol'shanskiĭ semigroup S is a non-degenerate representation of the involutive semigroup S (cf. Definition I.1(b)) for which the map $\pi: S \to B(\mathcal{H})$ is holomorphic. ■

Since we now work in the category of complex Ol'shanskiĭ semigroups and holomorphic representations we have to adapt the objects $\mathcal{A}(S)$ and $\mathcal{P}(S, \alpha)$ slightly to the new setting.

Definition II.4. Let V be a finite dimensional real vector space and V^* its dual space.
(a) For a subset $E \subseteq V$ we define its *dual set* by $E^\star := \{\alpha \in V^*: (\forall X \in E)\alpha(X) \geq 0\}$. Note that $E^\star$ is a closed convex cone in V^*.
(b) If $C \subseteq V$ is a convex set, then we define

$$\lim C := \{v \in V: v + C \subseteq C\} \qquad \text{and} \qquad B(C) := \{\alpha \in V^*: \inf \alpha(C) > -\infty\}$$

We call $\lim C$ the *limit cone* of C. Note that both $\lim C$ and $B(C)$ are convex sets and that $\lim C$ is closed if C is open or closed.
(c) If $C \subseteq V$ is a convex set, then we say C is *pointed* if C contains no affine lines, i.e., $\overline{\lim C} \cap -\overline{\lim C} = \{0\}$. ■

Lemma II.5. *Let $(\pi, \mathcal{H})$ be a holomorphic representation of S. Then $\|\alpha(s)\| := \|\pi(s)\|$ is a continuous $G \times G$-biinvariant absolute value on S and there exists a unique closed $\mathrm{Ad}(G)^*$-invariant convex subset $C \subseteq \mathfrak{g}^*$ with $-W^\star = \lim C$ such that*

$$\alpha(g \operatorname{Exp}(iX)) =: \alpha_C(g \operatorname{Exp}(iX)) := e^{\sup\langle X, C\rangle}$$

for $g \operatorname{Exp}(iX) \in S$.

Proof. [Ne99, Th. XI.3.5]. ■

Note that the condition $-W^\star = \lim C$ is equivalent to $-\overline{W} = \overline{B(C)}$ (cf. [Ne99, Lemma V.1.18]). We define now

$$\mathcal{A}_h(S) := \{\alpha_C: C \subseteq \mathfrak{g}^*, \text{ closed and convex, } -W^\star = \lim C, \mathrm{Ad}(G)^*.C = C\},$$

and for all $\alpha \in \mathcal{A}_h(S)$ we set

$$\mathcal{P}_h(S, \alpha) := \{\varphi \in \mathrm{Hol}(S): \varphi \text{ positive definite and } \alpha\text{-bounded}\}.$$

Finally we set $C_h^*(S, \alpha) := C^*(S, \alpha, \mathcal{P}_h(S, \alpha))$ (cf. Definition I.3).

Since each α is locally bounded, the canonical mapping $j: S \to C_h^*(S, \alpha)$ becomes holomorphic. The analog of Lemma I.4 now reads as follows.

Lemma II.6. *The prescription $\widetilde{\pi} \mapsto \pi := \widetilde{\pi} \circ j$ defines a bijection between non-degenerate representations $(\widetilde{\pi}, \mathcal{H})$ of $C_h^*(S, \alpha)$ and α-bounded holomorphic representations $(\pi, \mathcal{H})$ of S.* ■

Definition II.7. A C^*-algebra $\mathcal{A}$ is said to be *CCR* (*completely continuous representations*) or *liminal* if for all irreducible representations $(\pi, \mathcal{H})$ of $\mathcal{A}$ the image $\pi(\mathcal{A})$ is contained in the space of compact operators $\mathcal{K}(\mathcal{H})$ of $\mathcal{H}$. ■

Theorem II.8. (K.-H. Neeb) *If S is a complex Ol'shanskiĭ semigroup and $\alpha \in \mathcal{A}_h(S)$, then the C^*-algebra $C_h^*(S, \alpha)$ is CCR.*

Proof. [Ne99, Th. XI.6.6]. ■

The hull-kernel topology on the dual

In this subsection we introduce for each $\alpha \in \mathcal{A}_h(S)$ the hull-kernel topology on the set $\widehat{S}_\alpha$ of equivalence classes of irreducible α-bounded holomorphic representations of S. Then we characterize the hull-kernel topology on $\widehat{S}_\alpha$ in terms of the compact convergence of the corresponding matrix coefficients of irreducible holomorphic representations.

Lemma II.9. *On $\mathcal{P}_h(S,\alpha)$ the topology of pointwise convergence coincides with the topology of compact convergence.*

Proof. Let $\varphi_n \to \varphi$ pointwise on S and $K \subseteq S$ a compact subset. Then we find an element $t \in S$ and a compact subset $\widetilde{K} \subseteq S$ such that $K \subset t^* \widetilde{K} t$ holds (cf. [HiNe93, 3.19]). Since for all $\psi \in \mathcal{P}_h(S,\alpha)$ and $s,t \in S$ the inequality

$$|\psi(t^*st)| \leq \alpha(s)\psi(t^*t)$$

holds (cf. Definition I.2(d)), the locally boundedness of α implies that all functions in $\mathcal{P}_h(S,\alpha)$ are uniformly bounded on K. Thus by Montel's Theorem $\mathcal{P}_h(S,\alpha)|_K$ is a compact subset of $C(K)$. In particular, $\varphi_n \to \varphi$ pointwise on S implies $\varphi_n \to \varphi$ uniformly on K, as was to be shown. ∎

Theorem II.10. *If we equip $\mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ with the topology of compact convergence on S and $E(C^*_h(S,\alpha))$ with the weak-$*$-topology of $C^*_h(S,\alpha)$, then the mapping*

$$\Psi\colon \mathcal{P}_h(S,\alpha)_{\mathbf{1}} \to E(C^*_h(S,\alpha)), \quad \varphi \mapsto f_\varphi$$

is a homeomorphism.

Proof. In view of Lemma II.6 and Lemma II.9, this follows now from Theorem I.5. ∎

Definition II.11. (Hull-kernel topology) Let $\mathcal{A}$ be a C^*-algebra.
(a) A two-sided ideal J of $\mathcal{A}$ is called *primitive* if there exists an irreducible representation $(\pi,\mathcal{H})$ of $\mathcal{A}$ such that $J = \ker\pi$. The space of primitive ideals is denoted by $\mathrm{Prim}(\mathcal{A})$.
(b) For a subset $M \subseteq \mathrm{Prim}(\mathcal{A})$ we set

$$I(M) := \bigcap_{J\in M} J \qquad \text{and} \qquad \overline{M} := \{J \in \mathrm{Prim}(\mathcal{A}) : J \supset I(M)\}.$$

Then the prescription

$$\mathcal{T}_J := \{\mathrm{Prim}(\mathcal{A})\backslash\overline{M} : M \subset \mathrm{Prim}(\mathcal{A})\}$$

defines a topology on $\mathrm{Prim}(\mathcal{A})$ (cf. [Dix82, Sect. 3.1]), the so-called *Jacobson topology*.
(c) Let $\widehat{\mathcal{A}}$ denote the set of equivalence classes of irreducible representations of $\mathcal{A}$. For each irreducible representation $(\pi,\mathcal{H}_\pi)$ of $\mathcal{A}$ we denote by $[\pi]$ its equivalence class. Consider the map

$$\widehat{\mathcal{A}} \to \mathrm{Prim}(\mathcal{A}), \ [\pi] \mapsto \ker\pi.$$

If we equip $\mathrm{Prim}(\mathcal{A})$ with the Jacobson topology, then the initial topology on $\widehat{\mathcal{A}}$ under this map is called the *hull-kernel* or *Fell topology* on $\widehat{\mathcal{A}}$. ∎

Definition II.12. Let S be a complex Ol'shanskiĭ semigroup and $\alpha \in \mathcal{A}_h(S)$ an absolute value on it. We denote by $\widehat{S}$ the set of equivalence classes of irreducible holomorphic representations of S and by $\widehat{S}_\alpha$ the subset of all α-bounded ones. Note that $C^*_h(S,\alpha)\widehat{\ }$ can be identified with $\widehat{S}_\alpha$ (cf. Lemma II.6). The topology on $\widehat{S}_\alpha$ induced from the hull-kernel topology on $C^*_h(S,\alpha)\widehat{\ }$ is denoted by $\mathcal{T}^\alpha_{hk}$. Note that $\widehat{S}_\alpha$ is countable at infinity, since $C^*_h(S,\alpha)$ is separable. ∎

Remark II.13. (a) If $(\pi,\mathcal{H}_\pi)$ is an α-bounded holomorphic representation of S and $v \in \mathcal{H}_\pi$, then an element of $\mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ is defined by $\pi_{v,v}(s) := \langle \pi(s).v, v\rangle$, $s \in S$. Conversely, the Gelfand-Naimark-Segal correspondance between positive definite functions and matrix coefficients together with [Ne99, Prop. III.1.23] imply that

$$\mathcal{P}_h(S,\alpha)_{\mathbf{1}} = \{\pi_{v,v}\colon v \in \mathcal{H}_\pi, \ (\pi,\mathcal{H}_\pi)\ \alpha-\text{bounded}\}.$$

(b) Every holomorphic representation $(\pi,\mathcal{H})$ of S gives rise to a representation of the involutive semigroup $S \cup G$, also denoted by $(\pi,\mathcal{H})$, such that $\pi\,|_G$ is a uniquely determined unitary representation of G. Moreover, $\pi|_G$ is irreducible if and only if $\pi|_S$ is (cf. [Ne99, Ch. XI] for all that).

(c) In view of (a) and (b), we conclude that every $\varphi \in \mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ extends naturally to a positive definite function on $S \cup G$ which is continuous when restricted to G. In the sequel we identify the elements of $\mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ as positive definite functions on $S \cup G$. Further for all $\varphi \in \mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ all the *radial limits*

$$\lim_{\substack{t\to 0\\ t>0}} \varphi(g \operatorname{Exp}(itX)) = \varphi(g)$$

exist, where $g \in G$ and $X \in W$ (this is immediate from (a) and [Ne99, Prop. IV.3.2]). ∎

Theorem II.14. *For a subset M of $\widehat{S}_\alpha$ let $\overline{M}$ be the closure in the topology $\mathcal{T}_{hk}^\alpha$ (cf.* Definition II.12*). For $[\pi] \in \widehat{S}_\alpha$ and $v \in \mathcal{H}_\pi$ let $\pi_{v,v}(s) := \langle \pi(s).v, v\rangle$,*

$$E([\pi]) := \{\pi_{v,v} : v \in \mathcal{H}_\pi,\ \|v\| = 1\} \quad \text{and} \quad E(M) := \bigcup_{[\pi]\in M} E([\pi]).$$

Let $\overline{E(M)}$ be the closure of $E(M)$ in $\mathcal{P}_h(S,\alpha)_{\mathbf{1}}$, equipped with the topology of compact convergence on S. Then the following assertions are equivalent:

(1) $[\pi] \in \overline{M}$,

(2) $(\exists v \in \mathcal{H}_\pi,\ \|v\| = 1) \quad \pi_{v,v} \in \overline{E(M)}$,

(3) $(\forall v \in \mathcal{H}_\pi,\ \|v\| = 1) \quad \pi_{v,v} \in \overline{E(M)}$,

(4) $E([\pi]) \cap \overline{\operatorname{span} E(M)} \neq \{0\}$.

Proof. In view of Theorem II.10 and Remark II.13(a), the assertion follows now from [Dix82, Th. 3.4.10]. ∎

Comparison with the topology on $\widehat{G}$

Recall from Remark II.13(b) that every irreducible holomorphic representation of S defines in a unique manner an irreducible unitary representation of G. Thus we may consider $\widehat{S}$ as a certain subset of $\widehat{G}$ and the topology on $\widehat{G}$ gives rise to a topology on $\widehat{S}_\alpha$ which we denote by $\mathcal{T}_G^\alpha$.

In this subsection we compare the hull-kernel topology $\mathcal{T}_{hk}^\alpha$ on $\widehat{S}_\alpha$ with the topology $\mathcal{T}_G^\alpha$ induced from $\widehat{G}$. With real analysis methods, i.e., boundary values of Poisson transforms, we show that $\mathcal{T}_{hk}^\alpha \subseteq \mathcal{T}_G^\alpha$.

Lemma II.15. *Let $\alpha \in \mathcal{A}_h(S)$. Then for each compact subset $Q \subseteq W$ there exists a norm $\|\cdot\|$ on $\mathfrak{g}$ and a constant $c > 0$ sucht that*

$$(\forall X \in \mathbb{R}^+ Q) \qquad \alpha(\operatorname{Exp}(iX)) \leq c e^{\|X\|}.$$

Proof. If $0 \in W$, then $W = \mathfrak{g}$ and $\alpha = 1$ by Lemma II.5. In this case the assertion of the lemma is clear. Thus we may assume that $0 \notin W$.

Note that $\alpha = \alpha_C$ for some closed convex subset $C \subseteq \mathfrak{g}^*$ with $-\overline{W} = \overline{B(C)}$ by the definition of $\mathcal{A}_h(S)$. Thus the compactness of Q in W shows that

$$\sup\nolimits_{X\in[0,1]Q} \alpha(\operatorname{Exp}(iX)) =: c < \infty.$$

Let $\|\cdot\|$ be a norm on $\mathfrak{g}$ such that $\|X\| > 1$ for all $X \in]1,\infty[Q$ (this is possible since $0 \notin W$). Note that the mapping $W \to \mathbb{R}^+,\ X \mapsto \log\alpha(\operatorname{Exp}(iX))$ is positively homogeneous by the construction of $\mathcal{A}_h(S)$. Thus is $Y = kX \in \mathbb{R}^+ Q$ with $k \in \mathbb{R}^+$ and $X \in Q$, then

$$\alpha(\operatorname{Exp}(iY)) = \alpha(\operatorname{Exp}(iX))^k \leq c^k \leq c^{\|Y\|}.$$

Now an appropriate rescaling of $\|\cdot\|$ yields the assertion. ∎

Proposition II.16. *The mapping* $(\widehat{S}_\alpha, \mathcal{T}_G^\alpha) \to (\widehat{S}_\alpha, \mathcal{T}_{hk}^\alpha)$ *is continuous, i.e., we have* $\mathcal{T}_{hk}^\alpha \subset \mathcal{T}_G^\alpha$.

Proof. Recall from Remark II.13(c) that the elements of $\mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ are identified with the positive definite functions on $S \cup G$ which restrictions $\varphi|_S$ are holomorphic and α-bounded and have $\varphi|_G$ as continuous radial limit.

In view of [Dix82, Prop. 18.1.5] and Theorem II.14 it suffices to show that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ which satisfies

$$\varphi_n \to \varphi \quad \text{compactly on } G,$$

also satisfies

$$\varphi_n \to \varphi \ \text{ compactly on } S.$$

Let $K \subseteq S$ be a compact subset and $s = g\,\mathrm{Exp}(iX) \in K$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ which is contained in W and set $Q := \mathrm{conv}(\{X_1, \ldots, X_n\})$. We choose the basis in such a way that $X \in Q$. Since $d\,\mathrm{Exp}(iX)$ is everywhere invertible, we find an open subset $U \subset \mathfrak{g} + iQ$ with $iX \in U$ such that $\mathrm{Exp}|_U$ is a diffeomorphism. W.l.o.g. we may assume $U \subseteq Q$ and also $K = g\,\mathrm{Exp}(U)$. Let $\mathbb{R}_+ :=]0,\infty[$. Then the mapping

$$h\colon \Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n) \to S, \quad (z_1, \ldots, z_n) \mapsto g\,\mathrm{Exp}(z_1X_1 + \ldots + z_nX_n),$$

induces a map

$$h_* : \mathrm{Hol}(S) \to \mathrm{Hol}(\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n)), \quad \psi \mapsto \widetilde{\psi} := \psi \circ h.$$

Thus we only have to show that the compact convergence of $\widetilde{\varphi_n}|_{\mathbb{R}^n}$ implies the compact convergence of $\widetilde{\varphi_n}$ on the tube domain $\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n)$. Recall that each $\psi \in \mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ satisfies the estimate

$$(\forall s \in S) \quad |\psi(s)| \le \psi(\mathbf{1})\alpha(s) = \alpha(s)$$

(cf. Definition I.2(d)). Let $c > 0$ and define $f \in \mathrm{Hol}(\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n))$

$$f : \Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n) \to \mathbb{C}, \quad f(z) := ce^{-ic(z_1+\ldots+z_n)}.$$

In view of Lemma II.15, we then have $|\widetilde{\psi}(z)| \le |f(z)|$ for all $z \in \Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n)$ and $\psi \in \mathcal{P}_h(S,\alpha)_{\mathbf{1}}$ provided $c > 0$ is chosen sufficiently large. Since f has no zeros, we may replace $\widetilde{\varphi}_n$ by $\varphi_n' := \frac{1}{f}\widetilde{\varphi}_n$. Note that the functions φ_n', φ' are elements of $\mathrm{Hol}(\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n))$ uniformly bounded by 1. Further we replace φ_n', φ by $\varphi_n'' := g\varphi_n'$ and $\varphi'' := g\varphi'$, where $g\colon \Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n) \to \mathbb{C}$, $g(z) := \frac{1}{z_1+i} \cdot \ldots \cdot \frac{1}{z_n+i}$. Note that the functions φ_n'', φ'' are uniformly bounded elements of $\mathrm{Hol}(\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n))$ which are uniformly vanishing at infinity.

Let

$$p\colon \overline{\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n)} \to \mathbb{R}, \quad p(x_1 + iy_1, \ldots, x_n + iy_n) := \left(\frac{1}{\pi}\right)^n \prod_{j=1}^n \frac{y_j}{x_j^2 + y_j^2}$$

be the Poisson kernel of the upper half plane and

$$P\colon L^\infty(\mathbb{R}^n) \to \mathrm{Harm}(\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n)), \quad P(f)(z) := \int_{\mathbb{R}^n} f(x)p(z-x)\,dx$$

the corresponding Poisson transform. According to [SW75, §2, 2.1,5], the functions φ_n'', φ'' are the Poisson transforms of their boundary values, i.e., $\varphi_n'' = P(\varphi_n''|_{\mathbb{R}^n})$, $\varphi'' = P(\varphi''|_{\mathbb{R}^n})$ and we have

$$\|\varphi_n''\|_{\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n),\infty} = \|\varphi_n''\|_{\mathbb{R}^n,\infty} \tag{2.1}$$

and analogously for φ''. Since the functions $\varphi_n''|_{\mathbb{R}^n}$, $\varphi''|_{\mathbb{R}^n}$ vanish uniformly at infinity, equation (2.1) implies that the compact convergence of the $\varphi_n''|_{\mathbb{R}^n} \to \varphi''|_{\mathbb{R}^n}$ implies the compact convergence on $\Gamma_{\mathbb{R}^n}(\mathbb{R}_+^n)$. This concludes the proof of the proposition. ∎

Highest weight representations

In this section we take a closer look at the irreducible holomorphic representations of S. It turns out that they are obtained by analytic continuation of unitary highest weight representations of G.

Note that if a real Lie algebra admits a non-empty open elliptic convex cone, then there exists a compactly embedded Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ (cf. [Ne99, Th. VII.1.8]). To step further we first need some terminology concerning Lie algebras with compactly embedded Cartan subalgebras.

Definition II.17. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$ with compactly embedded Cartan subalgebra $\mathfrak{t}$.

(a) Associated to the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ in the complexification $\mathfrak{g}_\mathbb{C}$ there is a root decomposition as follows. For a linear functional $\alpha \in \mathfrak{t}_\mathbb{C}^*$ we set

$$\mathfrak{g}_\mathbb{C}^\alpha := \{X \in \mathfrak{g}_\mathbb{C} \colon (\forall Y \in \mathfrak{t}_\mathbb{C})\ [Y, X] = \alpha(Y)X\}$$

and write $\Delta := \{\alpha \in \mathfrak{t}_\mathbb{C}^* \setminus \{0\} \colon \mathfrak{g}_\mathbb{C}^\alpha \neq \{0\}\}$ for the set of roots. Then $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\mathbb{C}^\alpha$, $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ for all $\alpha \in \Delta$ and $\overline{\mathfrak{g}_\mathbb{C}^\alpha} = \mathfrak{g}_\mathbb{C}^{-\alpha}$, where $X \to \overline{X}$ denotes complex conjugation on $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{g}$.

(b) Let $\mathfrak{k}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Then a root α is said to be *compact* if $\mathfrak{g}_\mathbb{C}^\alpha \subseteq \mathfrak{k}_\mathbb{C}$ and *non-compact* otherwise. We write Δ_k for the set of compact roots and Δ_n for the non-compact ones.

(c) A positive system Δ^+ of roots is a subset of Δ for which there exists a regular element $X_0 \in i\mathfrak{t}^*$ with $\Delta^+ := \{\alpha \in \Delta \colon \alpha(X_0) > 0\}$. We call a positive system $\mathfrak{k}$-*adapted* if the set $\Delta_n^+ := \Delta_n \cap \Delta^+$ is invariant under the *Weyl group* $\mathcal{W}_\mathfrak{k} := N_{\mathrm{Inn}(\mathfrak{k})}(\mathfrak{t})/Z_{\mathrm{Inn}(\mathfrak{k})}(\mathfrak{t})$ acting on $\mathfrak{t}$. We recall from [Ne99, Prop. VII.2.14] that there exists a $\mathfrak{k}$-adapted positive system if and only if $\mathfrak{z}_\mathfrak{g}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$. In this case we say $\mathfrak{g}$ is *quasihermitian*. In this case it is easy to see that $\mathfrak{s}$ is quasihermitian, too, and so all simple ideals of $\mathfrak{s}$ are either compact or hermitian.

(d) We associate to the positive system Δ^+ the convex cones

$$C_{\min} := \overline{\mathrm{cone}\{i[\overline{X_\alpha}, X_\alpha] \colon X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha, \alpha \in \Delta_n^+\}},$$

and $C_{\max} := (i\Delta_n^+)^\star = \{X \in \mathfrak{t} \colon (\forall \alpha \in \Delta_n^+)\ i\alpha(X) \geq 0\}$. Note that both $C_{\min}$ and $C_{\max}$ are closed convex cones in $\mathfrak{t}$.

(e) Write $p_\mathfrak{t} \colon \mathfrak{g} \to \mathfrak{t}$ for the orthogonal projection along $[\mathfrak{t}, \mathfrak{g}]$ and set $\mathcal{O}_X := \mathrm{Inn}(\mathfrak{g}).X$ for the adjoint orbit through $X \in \mathfrak{g}$. We define the *minimal* and *maximal cone* associated to Δ^+ by

$$W_{\min} := \{X \in \mathfrak{g} \colon p_\mathfrak{t}(\mathcal{O}_X) \subseteq C_{\min}\} \quad \text{and} \quad W_{\max} := \{X \in \mathfrak{g} \colon p_\mathfrak{t}(\mathcal{O}_X) \subseteq C_{\max}\}$$

and note that both cones are convex, closed and $\mathrm{Inn}(\mathfrak{g})$-invariant. ■

From now on we assume that $\mathfrak{g}$ contains a compactly embedded Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and that there exists a non-empty open elliptic convex cone $W \subseteq \mathfrak{g}$. Then in view of [Ne99, Th. VII.3.8], there exits a $\mathfrak{k}$-adapted positive system Δ^+ such that

$$W_{\min} \subseteq \overline{W} \subseteq W_{\max}$$

holds, $W_{\max}^0$ is elliptic, $W_{\min} \cap \mathfrak{t} = C_{\min}$ and $W_{\max} \cap \mathfrak{t} = C_{\max}$. Recall that every elliptic $\mathrm{Ad}(G)$-invariant cone $W \subseteq \mathfrak{g}$ can be reconstructed by its intersection with $\mathfrak{t}$, i.e., $W = \mathrm{Ad}(G).(W \cap \mathfrak{t})$.

Definition II.18. Let Δ^+ be a positive system.

(a) For a $\mathfrak{g}_\mathbb{C}$-module V and $\beta \in (\mathfrak{t}_\mathbb{C})^*$ we write $V^\beta := \{v \in V : (\forall X \in \mathfrak{t}_\mathbb{C})X.v = \beta(X)v\}$ for the *weight space of weight* β and $\mathcal{P}_V = \{\beta \colon V^\beta \neq \{0\}\}$ for the set of weights of V.

(b) Let V be a $\mathfrak{g}_\mathbb{C}$-module and $v \in V^\lambda$ a $\mathfrak{t}_\mathbb{C}$-weight vector. We say that v is a *primitive element of* V (with respect to Δ^+) if $\mathfrak{g}_\mathbb{C}^\alpha.v = \{0\}$ holds for all $\alpha \in \Delta^+$.

(c) A $\mathfrak{g}_{\mathbb{C}}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.
(d) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We write K for the analytic subgroup of G corresponding to $\mathfrak{k}$. Let $(\pi, \mathcal{H})$ be a unitary representation of G. A vector $v \in \mathcal{H}$ is called *K-finite* if it is contained in a finite dimensional K-invariant subspace. We write $\mathcal{H}^{K,\omega}$ for the space of analytic K-finite vectors.
(e) An irreducible unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G is called a *highest weight representation* with respect to Δ^+ and highest weight $\lambda \in i\mathfrak{t}^*$ if $L(\lambda) := \mathcal{H}_\lambda^{K,\omega}$ is a highest weight module for $\mathfrak{g}_{\mathbb{C}}$ with respect to Δ^+ and highest weight λ. We write $HW(G, \Delta^+) \subset i\mathfrak{t}^*$ for the set of highest weights corresponding to unitary highest weight representations of G with respect to Δ^+. ■

Lemma II.19. *Let $S = \Gamma_G(W)$ be a complex Ol'shanskiĭ semigroup and Δ^+ be a $\mathfrak{k}$-adapted positive system with $C_{\min} \subseteq \overline{W} \cap \mathfrak{t} \subseteq C_{\max}$.*

(i) *If $(\pi, \mathcal{H})$ is an irreducible holomorphic representation of S, then $(\pi, \mathcal{H})$ extends to a representation of $S \cup G$, also denoted by $(\pi, \mathcal{H})$, such that $\pi|_G$ is a uniquely determined unitary highest weight representation of G with respect to Δ^+. Conversely, if $(\pi_\lambda, \mathcal{H}_\lambda)$ is a unitary highest weight representation of G with respect to Δ^+, then $(\pi_\lambda, \mathcal{H}_\lambda)$ extends to a uniquely determined holomorphic representation of S. In particular, the mapping*

$$HW(G, \Delta^+) \to \widehat{S}, \quad \lambda \mapsto [\pi_\lambda]$$

is bijective, hence gives a parametrization of $\widehat{S}$ by highest weights.

(ii) *If $\alpha = \alpha_C$ and HW_α denotes the subset of $HW(G, \Delta^+)$ corresponding to α-bounded representations, then*

$$HW_\alpha = \{\lambda \in HW(G, \Delta^+) : i\lambda \in C \cap \mathfrak{t}^*\}.$$

Proof. (i) This follows from [Ne99, Th. XI.2.3].
(ii) It follows from [Kr99a, Lemma IV.12] that

$$HW_\alpha = \{\lambda \in HW(G, \Delta^+) : (\forall X \in W \cap \mathfrak{t})\ e^{\lambda(iX)} \leq \alpha(\mathrm{Exp}(iX))\}.$$

We define $f : \mathfrak{t} \to \mathbb{R} \cup \{+\infty\}$, $X \mapsto \sup\langle X, C\rangle$ and note that f is a lower semicontinuous convex function. Since C was supposed to be closed, convex and $\mathrm{Ad}^*(G)$-invariant, it follows in particular that $f(X) = \sup\langle X, C \cap \mathfrak{t}^*\rangle$ for all $X \in \mathfrak{t}$ (cf. [Ne99,Prop. V.2.2]).

Fix $\lambda \in HW(G, \Delta^+)$. Then we have $\lambda \in HW_\alpha$ if and only if $i\lambda(X) \leq f(X)$ for all $X \in W \cap \mathfrak{t}$. Let $D_f := \{X \in \mathfrak{t} : f(X) < \infty\}$. Then $\overline{W} = -\overline{B(C)}$ implies that $D_f = (-B(C)) \cap \mathfrak{t} \subseteq \overline{W} \cap \mathfrak{t} = \overline{W \cap \mathfrak{t}}$. Thus it follows from [Ne99, Prop. V.3.2(i)] that $\lambda \in HW_\alpha$ if and only if $i\lambda \leq f$ on $\mathfrak{t}$, i.e., $i\lambda \in C \cap \mathfrak{t}^*$ as was to be shown. ■

The topology on $\widehat{S}_\alpha$

In view of Lemma II.19, we can parametrize $\widehat{S}_\alpha$ by the subset $HW_\alpha \subseteq i\mathfrak{t}^*$. The euclidean topology on HW_α thus gives rise to a topology on $\widehat{S}_\alpha$ which we denote by $\mathcal{T}_e^\alpha$.

In the sequel we will show that for generic absolute values α all the topologies $\mathcal{T}_{hk}^\alpha$, $\mathcal{T}_G^\alpha$ and $\mathcal{T}_e^\alpha$ coincide on $\widehat{S}_\alpha$.

Associated to the complex Lie subalgebras $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}_{\mathbb{C}}^+ := \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_{\mathbb{C}}^\alpha$ and $\mathfrak{p}_{\mathbb{C}}^- := \bigoplus_{\alpha \in -\Delta_n^+} \mathfrak{g}_{\mathbb{C}}^\alpha$ of $\mathfrak{g}_{\mathbb{C}}$ we have analytic subgroups $K_{\mathbb{C}}$, P^+ and P^- of $G_{\mathbb{C}}$. Recall from [Ne99, Ch. XII] that the groups $P^\pm$ are biholomorphic to $\mathfrak{p}^\pm$ via the exponential mapping and that the multiplication mapping

$$P^- \times K_{\mathbb{C}} \times P^+ \to G_{\mathbb{C}}, \quad (p_-, k, p_+) \mapsto p_- k p_+$$

is biholomorphic onto its open image $P^- K_{\mathbb{C}} P^+ \subseteq G_{\mathbb{C}}$. Set $S_1 := \Gamma_{G_1}(W) \subseteq G_{\mathbb{C}}$ and recall from [Ne99, Th. XII.4.6] that

$$G_1 \subset \overline{S_1} \subset P^- K_{\mathbb{C}} P^+.$$

Further, if $P^-\widetilde{K_{\mathbb{C}}}P^+ \cong P^- \times \widetilde{K_{\mathbb{C}}} \times P^+$ denotes the simply connected covering of $P^-K_{\mathbb{C}}P^+$, then the chain of inclusions from above lifts to

$$\widetilde{G} \subset \overline{\widetilde{S}} \subset P^-\widetilde{K_{\mathbb{C}}}P^+$$

(cf. [Ne99, Cor. XII.4.7]). We denote by

$$\kappa: P^-\widetilde{K_{\mathbb{C}}}P^+ \to \widetilde{K_{\mathbb{C}}}, \quad s \mapsto \kappa(s)$$

the holomorphic projection on the middle component.

If $(\pi_\lambda, \mathcal{H}_\lambda)$ is a unitary highest weight representation of G, then we denote by $v_\lambda \in \mathcal{H}_\lambda$ a normalized highest weight vector. Further we set $F(\lambda) := U(\mathfrak{k}_{\mathbb{C}}).v_\lambda$ for the minimal $\mathfrak{k}$-type and write $(\pi_\lambda^K, F(\lambda))$ for the finite dimensional holomorphic representation of $\widetilde{K_{\mathbb{C}}}$ with highest weight λ. Finally we denote by $HW(\Delta_k^+)$ the set of linear functionals on $\mathfrak{t}_{\mathbb{C}}^*$ which are dominant integral with respect to Δ_k^+ and note that $HW(G, \Delta^+) \subseteq HW(\Delta_k^+)$.

Proposition II.20. *The mapping*

$$\Phi: S \times HW(G, \Delta^+) \to \mathbb{C}, \quad (s, \lambda) \mapsto \psi_\lambda(s) := \langle \pi_\lambda(s).v_\lambda, v_\lambda \rangle$$

lifts to a continuous mapping $\widetilde{\Phi}: P^-\widetilde{K_{\mathbb{C}}}P^+ \times HW(\Delta_k^+) \to \mathbb{C}$ *which is holomorphic in the first variable and given explicitly by* $\widetilde{\Phi}(s, \lambda) = \langle \pi_\lambda^K(\kappa(s)).v_\lambda, v_\lambda \rangle$.

Proof. W.l.o.g. we may assume that $S = \widetilde{S}$. Fix $\lambda \in HW(G, \Delta^+)$. Since v_λ is an analytic vector of $(\pi_\lambda, \mathcal{H}_\lambda)$, we find a zero-neighborhood $U \subset \mathfrak{g}$ such that on $U_{\mathbb{C}} := U + iU \subset \mathfrak{g}_{\mathbb{C}}$ the series

$$\sum_{j=0}^{\infty} \frac{1}{j!} \langle d\pi_\lambda(X)^j.v_\lambda, v_\lambda \rangle \tag{2.2}$$

converges uniformly (cf. [Ne99, Lemma XI.2.1]). Note that the value of (2.2) coincides with $\psi_\lambda(\exp(X))$ for all $X \in U$.

Let $D \subseteq \mathfrak{g}_{\mathbb{C}}$ be the connected component of $\{0\}$ in $\exp_{G_{\mathbb{C}}}^{-1}(P^-K_{\mathbb{C}}P^+)$ and $\mathrm{Exp}: D \to P^-\widetilde{K_{\mathbb{C}}}P^+$ the lifting of $\exp_{G_{\mathbb{C}}}|_D$ satisfying $\mathrm{Exp}(0) = \mathbf{1}$. If we choose U sufficiently small, we may assume that $\mathrm{Exp}(U_{\mathbb{C}}) \subset P^-\widetilde{K_{\mathbb{C}}}P^+$ and that $\mathrm{Exp}|_{U_{\mathbb{C}}}$ is a diffeomorphism onto its image. Let V^-, V_0, V^+ in P^-, $\widetilde{K_{\mathbb{C}}}$, P^+ be connected open $\mathbf{1}$-neighborhoods with $V^-V_0V^+ \subset \mathrm{Exp}(U_{\mathbb{C}})$.

According to the the uniform convergence of (2.2) on $U_{\mathbb{C}}$, the prescription

$$\psi'_\lambda: \mathrm{Exp}(U_{\mathbb{C}}) \to \mathbb{C}, \quad \psi'_\lambda(\mathrm{Exp}(X)) := \sum_{j=0}^{\infty} \frac{1}{j!} \langle d\pi_\lambda(X)^j.v_\lambda, v_\lambda \rangle$$

defines a holomorphic function on $\mathrm{Exp}(U_{\mathbb{C}})$. We claim that

$$\psi_\lambda(s) = \psi'_\lambda(\kappa(s)) = \langle \pi_\lambda^K(\kappa(s)).v_\lambda, v_\lambda \rangle \quad \text{for all } s \in V^-V_0V^+. \tag{2.3}$$

In (2.3) the second equality holds by definition so that it remains to prove the first one. For each $Y \in \mathfrak{g}_{\mathbb{C}}$ let L_Y the left invariant vector field on $G_{\mathbb{C}}$ with $L_Y(\mathbf{1}) = Y$. The corresponding vectorfield on $V^-V_0V^+$ obtained by restriction and lifting is denoted by $\widetilde{L}_Y$. Note that the mapping

$$\widetilde{L}: \mathfrak{g}_{\mathbb{C}} \to \mathrm{Der}(C^\infty(V^-V_0V^+)), \quad X \mapsto \widetilde{L}_X$$

is a Lie algebra homomorphism and that all vectorfields $\widetilde{L}_Y$ are holomorphic. Let $Y = Y_1 + iY_2 \in \mathfrak{p}^+_{\mathbb{C}}$, $X \in U$ and $g := \exp(X) \in G$. Since v_λ is an analytic vector, we obtain that

$$\begin{aligned}(\widetilde{L}_Y.\psi'_\lambda)(\operatorname{Exp}(X)) &= (\widetilde{L}_{Y_1}.\psi'_\lambda)(\operatorname{Exp}(X)) + i(\widetilde{L}_{Y_2}.\psi'_\lambda)(\operatorname{Exp}(X))\\ &= \frac{d}{dt}\Big|_{t=0}\psi_\lambda(g\exp(tY_1)) + i\frac{d}{dt}\Big|_{t=0}\psi_\lambda(g\exp(tY_2))\\ &= \frac{d}{dt}\Big|_{t=0}\langle\pi_\lambda(g\exp(tY_1)).v_\lambda, v_\lambda\rangle + i\frac{d}{dt}\Big|_{t=0}\langle\pi_\lambda(g\exp(tY_2)).v_\lambda, v_\lambda\rangle\\ &= \frac{d}{dt}\Big|_{t=0}\langle\pi_\lambda(\exp(tY_1)).v_\lambda, \pi_\lambda(g^{-1}).v_\lambda\rangle + i\frac{d}{dt}\Big|_{t=0}\langle\pi_\lambda(\exp(tY_2)).v_\lambda, \pi_\lambda(g^{-1}).v_\lambda\rangle\\ &= \langle d\pi_\lambda(Y_1).v_\lambda, v_\lambda\rangle + i\langle d\pi_\lambda(Y_2).v_\lambda, v_\lambda\rangle = \langle d\pi_\lambda(Y).v_\lambda, \pi_\lambda(g^{-1}).v_\lambda\rangle = 0.\end{aligned}$$

Thus $(\widetilde{L}_Y.\psi'_\lambda)\mid_{\operatorname{Exp}(U)} = 0$ and therefore $\widetilde{L}_Y.\psi'_\lambda = 0$ by the Identity Theorem for analytic functions. Therefore ψ'_λ is constant on all integral curves of $\widetilde{L}_Y$, i.e.,

$$\psi'_\lambda(p_-kp_+) = \psi'_\lambda(p_-k) \quad \text{for all } p_-kp_+ \in V^-V_0V^+.$$

Similarly one shows that ψ'_λ is constant on the "right cosets" of V^-, concluding the proof of (2.3).

It follows from (2.3) that ψ_λ extends to a holomorphic function

$$\widetilde{\psi}_\lambda : P^-\widetilde{K_{\mathbb{C}}}P^+ \to \mathbb{C}, \quad s \mapsto \langle\pi^K_\lambda(\kappa(s)).v_\lambda, v_\lambda\rangle.$$

In view of the Cartan-Weyl-Theorem of Highest Weight for finite dimensional representations of complex reductive Lie algebras, we have $HW(\Delta^+_k) = \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})^* + \Gamma$, where Γ is an additive discrete subsemigroup of $i(\mathfrak{t}\cap[\mathfrak{k},\mathfrak{k}])^*$. Thus the continuity of $\widetilde{\Phi}$ reduces to showing continuity of the maps

$$\widetilde{\Phi}_\gamma : \widetilde{K_{\mathbb{C}}} \times \big(\mathfrak{z}(\mathfrak{k}_{\mathbb{C}})^* + \gamma\big) \to \mathbb{C}, \quad (k,\lambda) \mapsto \langle\pi^K_\lambda(k).v_\lambda, v_\lambda\rangle,$$

where $\gamma \in \Gamma$. Note that $\widetilde{K_{\mathbb{C}}} \cong \mathfrak{z}(\mathfrak{k}_{\mathbb{C}}) \times [\widetilde{K_{\mathbb{C}}}, \widetilde{K_{\mathbb{C}}}]$ by the simple connectedness of $\widetilde{K_{\mathbb{C}}}$. Therefore an irreducible representation of $\widetilde{K_{\mathbb{C}}}$ correspoding to $z + \gamma \in \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})^* + \gamma$ is a tensor product representation of a one-dimensional representation of $\mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ with infinitesimal character z and a highest weight representation of $[\widetilde{K_{\mathbb{C}}}, \widetilde{K_{\mathbb{C}}}]$ corresponding to γ. This proves the continuity of the maps $\widetilde{\Phi}_\gamma$ and completes the proof of the proposition. ■

Corollary II.21. *The mappings $HW_\alpha \to (\widehat{S}_\alpha, \mathcal{T}^\alpha_{hk})$ and $HW_\alpha \to (\widehat{S}_\alpha, \mathcal{T}^\alpha_G)$ are continuous.*

Proof. Recall that $\widehat{S}_\alpha$ has a countable base (cf. Definition II.12). Thus by Theorem II.14 and [Dix82, Prop. 18.1.5], it suffices to show

$$\lambda_n \to \lambda \qquad \text{in the euclidean topology}$$

implies

$$\psi_{\lambda_n} \to \psi_\lambda \qquad \text{compactly on } S \text{ and } G.$$

But this is immediate from Proposition II.20. ■

Now we are going to prove that $\mathcal{T}^\alpha_e \subseteq \mathcal{T}^e_{hk}$. We start with a lemma which reduces the assertion to contraction representations.

Lemma II.22. (Reduction to Contractions) *Let $\alpha \in \mathcal{A}_h(S)$. Set $\mathfrak{g}^\sharp := \mathfrak{g} \oplus \mathbb{R}$ and $G^\sharp = G \times \mathbb{R}$. Then the following assertions hold:*

(i) *The prescription*

$$W^\sharp := \{(X,t) \in W \times \mathbb{R}^+ : \alpha(\operatorname{Exp}(iX)) < e^t\}.$$

defines an open convex $\mathrm{Ad}(G^\sharp)$*-invariant elliptic cone in* $\mathfrak{g}^\sharp$ *and a complex Ol'shanskiĭ semigroup* $S^\sharp := \Gamma_{G^\sharp}(W^\sharp) \subseteq S \times \mathbb{C}$ *can be build (cf.* Definition II.2*).*

(ii) *If* $(\pi, \mathcal{H})$ *is an* α*-bounded holomorphic representation of* S*, then* π *induces via* $\pi^\sharp(s,z) = e^{iz}\pi(s)$ *a* $\mathbf{1}$*-bounded holomorphic representation of* $S^\sharp$*. Moreover, the prescription* $\pi \mapsto \pi^\sharp$ *defines a bijection between* $\widehat{S}_\alpha$ *and* $\widehat{S^\sharp}_{\mathbf{1}}$*.*

(iii) *The* C^**-algebras* $C^*_h(S,\alpha)$ *and* $C^*_h(S^\sharp, \mathbf{1})$ *are isomorphic.*

Proof. (i) This is immediate from the definition of $\mathcal{A}_h(S)$.
(ii) From the construction of $S^\sharp$ and $\pi^\sharp$, it is clear that $\pi^\sharp$ is contractive, whenever π is α-bounded. Finally, recall from Lemma II.19(i) that every irreducible holomorphic representation of S, resp. $S^\sharp$, extends to a holomorphic representation of $\Gamma_G(W^0_{\max})$, resp. $\Gamma_{G^\sharp}(W^0_{\max} \oplus \mathbb{R})$. In view of this, the second assertion is also clear.
(iii) In the definition of $C^*_h(S,\alpha)$ (cf. Definition I.3) we used the full cone $\mathcal{P}_h(S,\alpha)$ of α-bounded positive definite functions. However, in view of Segal's Theorem (cf. [Dix82, Lemma 2.10.1]), we may replace $\mathcal{P}_h(S,\alpha)$ by the subcone of extremal generators $\mathrm{Ext}(\mathcal{P}_h(S,\alpha))$. Now the assertion follows from (i), Lemma II.6 and the construction of $C^*_h(S,\alpha)$. ■

A Lie group G is called a *(CA)-Lie group* (*closed adjoint*) if $\mathrm{Ad}(G)$ is closed in $\mathrm{Aut}(\mathfrak{g})$. Note that all reductive and nilpotent Lie groups are (CA)-Lie groups.

Lemma II.23. *Let* $([\pi_{\lambda_n}])_{n\in\mathbb{N}}$ *be a convergent sequence in* $(\widehat{S}_\alpha, \mathcal{T}^\alpha_{hk})$ *and* $[\pi_{\lambda_0}]$ *a limit point of it. Then the following assertions hold:*

(i) *The set* $\{\lambda_n : n \in \mathbb{N}\}$ *is relatively compact in* $i\mathfrak{t}^*$*. In particular, there exists a convergent subsequence* (λ_{n_k}) *with limit, say* λ_0'*.*

(ii) *If, in addition,* G *is a (CA)-Lie group, then we have*

$$\lambda_0' = \lambda_0 + \mu_0$$

for some $\mu_0 \in \mathbb{N}_0[\Delta^+]$*, where* $\mathbb{N}_0[\Delta^+]$ *denotes the additive subsemigroup of* $i\mathfrak{t}^*$ *generated by* $\Delta^+ \cup \{0\}$*.*

Proof. (i) W.l.o.g. we may assume that S is simply connected. In view of Lemma II.22, we may also assume that $\alpha \le \mathbf{1}$.

If $[\pi_{\lambda_n}] \to [\pi_{\lambda_0}]$ in $(\widehat{S}_\alpha, \mathcal{T}^\alpha_{hk})$, then Theorem II.14, implies in particular that there exists unit vectors $v_n \in \mathcal{H}_{\lambda_n}$, $n \in \mathbb{N}$, and a normalized highest weight vector v_0 of $\mathcal{H}_{\lambda_0}$ such that

$$\langle \pi_{\lambda_n}(s).v_n, v_n\rangle \to \langle \pi_{\lambda_0}(s).v_0, v_0\rangle \qquad \text{compactly on } S. \tag{2.4}$$

For each $\lambda \in HW(G,\Delta^+)$ we write $\mathcal{P}_\lambda := \mathcal{P}_{L(\lambda)}$ for the corresponding set of $\mathfrak{t}_\mathbb{C}$-weights (cf. Definition II.18(a)). Note that $\mathcal{P}_\lambda \subseteq \lambda - \mathbb{N}_0[\Delta^+]$ since $(\pi_\lambda, \mathcal{H}_\lambda)$ is a highest weight representation with respect to Δ^+ and highest weight λ.

Let $\mathfrak{t}^+ := \{X \in \mathfrak{t} : (\forall \alpha \in \Delta^+)\ i\alpha(X) > 0\}$ and let $X \in W \cap \mathfrak{t}^+$. Then we have for all $\lambda \in HW(G,\Delta^+)$ and $\mu_\lambda \in \mathcal{P}_\lambda$ that $\mu_\lambda(iX) \le \lambda(iX)$. We show that

$$\sup\nolimits_{n\in\mathbb{N}_0} \lambda_n(iX) \le 0 \quad \text{and} \quad \inf\nolimits_{n\in\mathbb{N}_0} \lambda_n(iX) > -\infty. \tag{2.5}$$

The first assertion in (2.5) is immediate from our reduction to $\alpha \le 1$. If the second assertion were false, we would find a subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ of $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \lambda_{n_k}(iX) = -\infty$. But this implies that

$$0 \le \lim_{k\to\infty} \langle \pi_{\lambda_{n_k}}(\mathrm{Exp}(iX)).v_{n_k}, v_{n_k}\rangle \le \lim_{k\to\infty} e^{\lambda_{n_k}(iX)} = 0.$$

In view of (2.4), this means that $e^{\lambda_0(iX)} = \langle \pi_{\lambda_0}(\mathrm{Exp}(iX)).v_0, v_0\rangle = 0$; a contradiction, completing the proof of (2.5).

By the definition of $\mathcal{A}_h(S)$ we have $\alpha = \alpha_C$ for some closed convex subset $C \subseteq \mathfrak{g}^*$ with $\overline{B(C)} = -\overline{W}$. In particular we have $X \in -\operatorname{int} B(C)$. Since $\lim C = -W^\star$, we see that C is pointed (cf. Definition II.4(c)), and so the evaluation mapping in X

$$\mathrm{ev}_X : C \to \mathbb{R}, \quad \lambda \mapsto -\lambda(X)$$

is proper (cf. [Ne99, Cor. V.1.16]). In view of Lemma II.19(ii), we have $HW_\alpha \subseteq -iC$, and so the properness of ev_X together with (2.5) imply (i).
(ii) For $\lambda \in HW(G, \Delta^+)$ and $v \in \mathcal{H}_\lambda$ we write $v = \sum_{\mu \in \mathcal{P}_\lambda} v^\mu$ for the Fourier series of v with respect to the $\mathfrak{t}_\mathbb{C}$-action. For each $n \in \mathbb{N}$ and $\mu \in \mathbb{N}_0[\Delta^+]$ we set $a_{n,\mu} := \langle v_n^{\lambda_n - \mu}, v_n^{\lambda_n - \mu} \rangle$. It follows from (2.4) that

$$\langle \pi_{\lambda_n}(\mathrm{Exp}(X)).v_n, v_n \rangle \to \langle \pi_{\lambda_0}(\mathrm{Exp}(X)).v_0, v_0 \rangle \quad \text{compactly on } \mathfrak{t} + i(W \cap \mathfrak{t}^+).$$

Considering the corresponding Fourier series, this means that

$$e^{\lambda_n(X)} \sum_{\mu \in \mathbb{N}_0[\Delta^+]} a_{n,\mu} e^{-\mu(X)} \to e^{\lambda_0(X)}$$

converges compactly on $\mathfrak{t} + i(W \cap \mathfrak{t}^+)$ or, equivalently, that

$$\sum_{\mu \in \mathbb{N}_0[\Delta^+]} a_{n,\mu} e^{-\mu(X)} \to e^{(\lambda_0 - \lambda_0')(X)} \quad \text{compactly on } \mathfrak{t} + i(W \cap \mathfrak{t}^+). \tag{2.6}$$

Set $T := \exp(\mathfrak{t})$ and note that $\mathrm{Ad}(T)$ is a compact group since G was supposed to be a (CA)-Lie group (cf. [Ne99, Sect. VII.1]). Thus (2.6) together with the Peter-Weyl Theorem applied to the action of $\mathrm{Ad}(T)$ on the Fréchet space $C^\infty(\mathfrak{t} + i(W \cap \mathfrak{t}^+))$ implies that $a_{n,\mu} \to 0$ except for $\mu = \mu_0 := \lambda_0' - \lambda_0$. This completes the proof of (ii). ■

For $\lambda \in HW(G, \Delta^+)$ we also consider the $\mathfrak{k}_\mathbb{C}$-module $F(\lambda)$ as $\mathfrak{k}_\mathbb{C} + \mathfrak{p}^+$-module with trivial $\mathfrak{p}^+$-action. Further we associate to λ the *generalized Verma module*

$$N(\lambda) := \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{k}_\mathbb{C} + \mathfrak{p}^+)} F(\lambda).$$

Note that $L(\lambda)$ is the unique irreducible quotient of the $\mathcal{U}(\mathfrak{g}_\mathbb{C})$-module $N(\lambda)$ (cf. [Ne99, Sect. IX.1]).

Definition II.24. Let $\alpha \in \mathcal{A}_h(S)$ be an absolute value for S. Then we call α *generic* if $L(\lambda) = N(\lambda)$ holds for all $\lambda \in HW_\alpha$. ■

Proposition II.25. *If G is a (CA)-Lie group and $\alpha \in \mathcal{A}_h(S)$ is generic, then the mapping*

$$(\widehat{S}_\alpha, \mathcal{T}_{hk}^\alpha) \to HW_\alpha, \quad [\pi_\lambda] \mapsto \lambda$$

is continuous, i.e., $\mathcal{T}_e^\alpha \subseteq \mathcal{T}_{hk}^\alpha$.

Proof. We use the notaion of Lemma II.23 and its proof. Let $[\pi_{\lambda_n}] \to [\pi_{\lambda_0}]$ in $(\widehat{S}_\alpha, \mathcal{T}_{hk}^\alpha)$. We have to show that $\lambda_n \to \lambda_0$. By Lemma II.23(i) we may assume that $(\lambda_n)_{n \in \mathbb{N}}$ converges to λ_0'. By Lemma II.23(ii) and its proof we find $\mu_0 \in \mathbb{N}_0[\Delta^+]$ with $\lambda_0' = \lambda_o + \mu_0$, unit vectors $v_n \in L(\lambda_n)^{\lambda_n - \mu_0}$, $n \in \mathbb{N}$, and a normalized highest weight vector v_0 of $L(\lambda_0)$ such that (2.4) holds.

To complete the proof of the proposition, we have to show that $\mu_0 = 0$. Let $Y \in \mathfrak{g}_\mathbb{C}^\alpha$ for some $\alpha \in \Delta^+$. After multiplying Y with a small non-zero scalar, we may assume that there exists an open subset $U \subseteq \mathfrak{t} + i(W \cap \mathfrak{t}^+)$ such that $V = \mathrm{Exp}(-\overline{Y}) \,\mathrm{Exp}(U) \,\mathrm{Exp}(Y) \subseteq P^- \widetilde{K_\mathbb{C}} P^+ \cap S$. Now it follows from (2.4)that

$$\begin{aligned}
&\langle \pi_{\lambda_n}(\mathrm{Exp}(-\overline{Y}) \,\mathrm{Exp}(X) \,\mathrm{Exp}(Y)).v_n, v_n \rangle = \langle \pi_{\lambda_n}(\mathrm{Exp}(X) \,\mathrm{Exp}(Y)).v_n, \pi_{\lambda_n}(\mathrm{Exp}(Y)).v_n \rangle \\
&= \sum_{k=0}^\infty e^{(\lambda_n - \mu_0 + k\alpha)(X)} \frac{1}{(k!)^2} \langle d\pi_{\lambda_n}(Y)^k.v_n, d\pi_{\lambda_n}(Y)^k.v_n \rangle \\
&\to \langle \pi_{\lambda_0}(\mathrm{Exp}(-\overline{Y}) \,\mathrm{Exp}(X) \,\mathrm{Exp}(Y)).v_0, v_0 \rangle = e^{\lambda_0(X)}
\end{aligned} \tag{2.7}$$

converges compactly on U. Again by the Peter-Weyl Theorem (cf. the proof of Lemma II.23(ii)) we deduce that

$$(2.8) \qquad (\forall Y \in \mathfrak{n} := \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{\mathbb{C}}^{\alpha}) \quad \lim_{n\to\infty} \|d\pi_{\lambda_n}(Y).v_n\| = 0.$$

As $\lambda_n \to \lambda_0'$ we may assume that $\lambda_n|_{\mathfrak{t}_{\mathbb{C}}\cap[\mathfrak{k}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}]} = \lambda_0'|_{\mathfrak{t}_{\mathbb{C}}\cap[\mathfrak{k}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}]}$ for all $n \in \mathbb{N}$. Set $\Lambda := \{\lambda_n : n \in \mathbb{N}\} \cup \{\lambda_0'\}$ and note that $\Lambda \subseteq HW_\alpha$ since $HW(G,\Delta^+)$ is closed in $i\mathfrak{t}^*$ (cf. [Kr99a, Sect. IV]) and HW_α is closed in $HW(G,\Delta^+)$ (cf. Lemma II.19(ii)).

As α is generic, [Kr99b] implies that we can identify all $L(\lambda) = N(\lambda)$, $\lambda \in \Lambda$, with $L(\lambda_1)$ as $[\mathfrak{k}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}] + \mathfrak{p}^+$-modules. Within this identification all operators $d\pi_\lambda(Y)$, $\lambda \in \Lambda$, coincide for $Y \in \mathfrak{p}^+$. Further the scalar products of the various $L(\lambda)$, $\lambda \in \Lambda$, define a family of inner products $(\langle\cdot,\cdot\rangle_\lambda)_{\lambda\in\Lambda}$ on $L(\lambda_1)$. Also from [Kr99b] we can deduce that for all $v \in L(\lambda_1)$ the mapping

$$\Lambda \to]0,\infty[, \quad \lambda \mapsto \langle v, v\rangle_\lambda$$

is continuous. In particular we may assume that $(v_n)_{n\in\mathbb{N}}$ converges in $L(\lambda_1)^{\lambda_1-\mu_0}$ with limit $v_0' \neq 0$. Thus (2.8) gives for all $Y \in \mathfrak{n}$

$$\|d\pi_{\lambda_0'}(Y).v_0'\| = \|d\pi_{\lambda_1}(Y).v_0'\|_{\lambda_0'} = \lim_{n\to\infty} \|d\pi_{\lambda_1}(Y).v_n\|_{\lambda_n} = \lim_{n\to\infty} \|d\pi_{\lambda_n}(Y).v_n\| = 0,$$

where the subscripts at the various norms indicate that we have identified the corresponding $L(\lambda)$ with $L(\lambda_1)$. Thus $v_0 \in L(\lambda_0')^{\lambda_0'-\mu_0}$ is a primitive element and so $\mu_0 = 0$ as was to be shown. ■

We now give an example that Proposition II.25 becomes false if α is non generic.

Example II.26. Let $G := \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ be the universal covering group of $\mathrm{Sl}(2,\mathbb{R})$. We choose

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a basis for $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{R})$. Then $\mathfrak{t} := \mathbb{R}U$ is a compactly embedded Cartan subalgebra. Let $\alpha \in i\mathfrak{t}^*$ be defined by $\alpha(U) = -2i$. The root system of $\mathfrak{g}$ is given by $\Delta = \{\pm\alpha\}$ with root spaces $\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathbb{C}(T + iH)$ and $\mathfrak{g}_{\mathbb{C}}^{-\alpha} = \mathbb{C}(T - iH)$. We define a positive system by $\Delta^+ := \{\alpha\}$. We denote by κ the Cartan-Killing form of $\mathfrak{g}$. Then the upper light cone

$$W := \{X = uU + tT + hH : u \geq 0,\ \kappa(X,X) \leq 0\} = \{X = uU + tT + hH : u \geq 0,\ h^2 + t^2 - u^2 \leq 0\}$$

is an invariant pointed cone in $\mathfrak{g}$. Moreover, W is up to sign the unique invariant elliptic cone in $\mathfrak{g}$ (cf. [HiNe93, Th. 7.25]). Let $S := \Gamma_G(W)$ be the complex Ol'shanskiĭ semigroup corresponding to G and W.

In the following we identify $\mathfrak{t}_{\mathbb{C}}^*$ with $\mathbb{C}$ via the isomorphism $\mathfrak{t}_{\mathbb{C}}^* \to \mathbb{C}$, $\lambda \mapsto \lambda(iU)$. Then $HW(G,\Delta^+) =]-\infty, 0]$.

If $\alpha \in \mathcal{A}_h(S)$ is an absolute value, then Lemma II.19(ii) implies that $HW_\alpha =]-\infty, t_\alpha]$ for some $t_\alpha \leq 0$. Conversely it follows from [Ne99, Th. VIII.3.21] that for each $t_\alpha \leq 0$ there exists an $\alpha \in \mathcal{A}_h(S)$ with $HW_\alpha =]-\infty, t_\alpha]$. Thus $\mathcal{A}_h(S) \to]-\infty, 0]$, $\alpha \mapsto t_\alpha$ is bijective, hence gives a parametrization of $\mathcal{A}_h(S)$ with non-positive real numbers. The generic absolute values correspond to $]-\infty, 0[$ and the only non-generic one is $\alpha = \mathbf{1}$ which corresponds to $t_\alpha = 0$ (cf. [Kr98, Ex. III.7]).

Let us now show that Proposition II.25 becomes false for the non-generic absolute value $\alpha = \mathbf{1}$. We choose $\lambda_n = -\frac{1}{n}$, $n \in \mathbb{N}$, and show that $[\pi_{-\frac{1}{n}}] \to [\pi_{-1}]$ in $(\widehat{S}_{\mathbf{1}}, \mathcal{T}_{hk}^{\mathbf{1}})$. Since we evidently have $[\pi_{-\frac{1}{n}}] \to [\pi_0]$, this will give us the non-continuity of the map in Proposition II.25.

For each $n \in \mathbb{N}$ Let $v_n \in L(-\frac{1}{n})^{-\frac{1}{n}-2} = L(\lambda_n)^{\lambda_n-\alpha}$, $n \in \mathbb{N}$, be a normalized vector. Further let $v_{-1} \in L(-1)$ be a normalized highest weight vector. Then by Theorem II.14 we will have $[\pi_{-\frac{1}{n}}] \to [\pi_{-1}]$ in $(\widehat{S}_{\mathbf{1}}, \mathcal{T}_{hk}^{\mathbf{1}})$ if $\langle \pi_{-\frac{1}{n}}(s).v_n, v_n\rangle \to \langle \pi_{-1}(s).v_{-1}, v_{-1}\rangle$ holds uniformly on compact subsets $U \subseteq S$.

Set $E := T + iH \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$. Now for $U \subseteq S$ compact we find an $R > 0$ such that $U \subseteq V^- W V^+$ with $W \subseteq \widetilde{K_{\mathbb{C}}}$ compact, $V^- = \{\mathrm{Exp}(u\overline{E}) : u \in \mathbb{C},\ |u| \leq R\}$ and $V^+ = \{\mathrm{Exp}(wE) : w \in \mathbb{C},\ |w| \leq R\}$. Further each $s \in S$ can uniquely be written as $s = \mathrm{Exp}(-u(s)\overline{E})\kappa(s)\,\mathrm{Exp}(w(s)E)$. Thus we get that

$$\begin{aligned}\langle \pi_{-\frac{1}{n}}(s).v_n, v_n\rangle &= \langle \pi_{-\frac{1}{n}}(\mathrm{Exp}(-u(s)\overline{E})\kappa(s)\,\mathrm{Exp}(w(s)E)).v_n, v_n\rangle \\ &= \kappa(s)^{\lambda_n - \alpha} + w(s)\overline{u(s)}\langle d\pi_{-\frac{1}{n}}(E).v_n, d\pi_{-\frac{1}{n}}(E).v_n\rangle.\end{aligned} \tag{2.9}$$

Now the formulas in [La85, Ch. VI] show that

$$\langle d\pi_{-\frac{1}{n}}(E).v_n, d\pi_{-\frac{1}{n}}(E).v_n\rangle = \frac{\left(\frac{1}{n}\right)^2}{\left(2 + \frac{1}{n}\right)} \to 0. \tag{2.10}$$

Thus we conclude from (2.9) and (2.10) that

$$\langle \pi_{-\frac{1}{n}}(s).v_n, v_n\rangle \to \kappa(s)^{-\alpha} = \langle \pi_{-1}(s).v_{-1}, v_{-1}\rangle$$

uniformly on U as was to be shown. ■

The Borel structure on the dual

In this subsection we investigate the Borel structures on $\widehat{S}_{\alpha}$ induced from our various topolgies on the dual.

Lemma II.27. *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}_+ := \{y \in \mathcal{A} : y \geq 0\}$. Then the mapping*

$$\widehat{\mathcal{A}} \to \mathbb{R}, \quad [\pi] \mapsto \sup \mathrm{Spec}(\pi(x)) = \|\pi(x)\|$$

lower semicontinuous.

Proof. [Dix82, Prop. 3.3.2]. ■

Proposition II.28. *Let $\mathcal{B}_{hk}^{\alpha}$ be the Borel structure on $\widehat{S}_{\alpha}$ induced from $\mathcal{T}_{hk}^{\alpha}$ and similarily $\mathcal{B}_{e}^{\alpha}$ the one induced from $\mathcal{T}_{e}^{\alpha}$. Then the mapping $(\widehat{S}_{\alpha}, \mathcal{B}_{hk}^{\alpha}) \to (\widehat{S}_{\alpha}, \mathcal{B}_{e}^{\alpha})$ is measurable, i.e., $\mathcal{B}_{e}^{\alpha} \subseteq \mathcal{B}_{hk}^{\alpha}$.*

Proof. Let $X \in W \cap \mathfrak{t}^+$. Then $\mathrm{Exp}(iX)$ is a symmetric element of S and so $j(\mathrm{Exp}(iX)) = j(\mathrm{Exp}(i\frac{1}{2}X))^2$ is positve in $C_h^*(S, \alpha)$. Let $v_\lambda \in \mathcal{H}_\lambda$ denote a normalized highest weight vector. Since $\mathcal{P}_\lambda \subset \lambda - \mathbb{N}_0[\Delta^+]$ we get

$$\begin{aligned}\sup \mathrm{Spec}(\widetilde{\pi_\lambda}(j(\mathrm{Exp}(iX))) &= \sup \mathrm{Spec}(\pi_\lambda(\mathrm{Exp}(iX))) \\ &= \langle \pi_\lambda(\mathrm{Exp}(iX)).v_\lambda, v_\lambda\rangle = e^{i\lambda(X)}.\end{aligned}$$

In view of Lemma II.27, for all $\beta \in \mathbb{R}$ the subsets

$$\begin{aligned}I_{X,\beta} :=& \{\lambda \in \widehat{S}_\alpha : e^{i\lambda(X)} \leq e^{\beta}\} \\ =& \{\lambda \in \widehat{S}_\alpha : i\lambda(X) \leq \beta\}\end{aligned}$$

are closed in $(\widehat{S}_\alpha, \mathcal{T}_{hk}^{\alpha})$. Now the assertion of the proposition follows since the system $\{I_{X,\beta} : X \in W \cap \mathfrak{t}^+,\ \beta \in \mathbb{R}\}$ generates the euclidean Borel structure on $\widehat{S}_\alpha$. ■

The main results

Theorem II.29. (The topologies on $\widehat{S}_\alpha$) *Let $S = \Gamma_G(W)$ be a complex Ol'shanskiĭ semigroup, $\alpha \in \mathcal{A}_h(S)$ an absolute value on S and $\widehat{S}_\alpha$ the equivalence classes of all α-bounded irreducible holomorphic representations of S. Let $\mathcal{T}_{hk}^\alpha$ denote the hull-kernel topology on $\widehat{S}_\alpha$ obtained from $C_h^*(S,\alpha)$, further $\mathcal{T}_G^\alpha$ the topology induced from $\widehat{G}$ and $\mathcal{T}_e^\alpha$ the euclidean topology obtained from the parametrization with heighest weights, and write $\mathcal{B}_{hk}^\alpha$, $\mathcal{B}_G^\alpha$ and $\mathcal{B}_e^\alpha$ for the corresponding Borel structures.*

(i) *We have*

$$\mathcal{T}_{hk}^\alpha \subseteq \mathcal{T}_G^\alpha \subseteq \mathcal{T}_e^\alpha \quad \text{and} \quad \mathcal{B}_{hk}^\alpha = \mathcal{B}_G^\alpha = \mathcal{B}_e^\alpha.$$

(ii) *If, in addition, G is a (CA)-Lie group and α is generic, then*

$$\mathcal{T}_{hk}^\alpha = \mathcal{T}_G^\alpha = \mathcal{T}_e^\alpha.$$

Proof. (i) The inclusion for the topologies follows from Proposition II.16 and Corollary II.21. Finally, the identity for the Borel structures follows from the chain of inclusions for the topologies together with Proposition II.28.
(ii) In view of (i), this follows from Proposition II.25. ■

Now we give two applications of Theorem II.24 to the structure of $C_h^*(S,\alpha)$ and the abstract representation theory of complex Ol'shanskiĭ semigroups. In the following two statements we use the language of [Dix82].

Corollary II.30. *Let G be a (CA)-Lie group, α a generic absolute value of S and $\mathfrak{A}_\alpha$ the C^*-algebra defined by the continuous field $\big(\mathcal{K}(\mathcal{H}_\lambda)\big)_{\lambda \in HW_\alpha}$ of elementary C^*-algebras. Then the mapping*

$$C_h^*(S,\alpha) \to \mathfrak{A}_\alpha, \quad x \mapsto (\widetilde{\pi}_\lambda(x))_{\lambda \in HW_\alpha}$$

(cf. Lemma II.6*) is an isomorphism of C^*-algebras.*

Proof. In view of Theorem II.8 and Theorem II.29(ii), $C_h^*(S,\alpha)$ is a CCR C^*-algebra with Hausdorff spectrum, and so the assertion follows from [Dix82, Th. 10.5.4]. ■

Recall the notion of multiplicity free representations: A holomorphic representation $(\pi,\mathcal{H})$ of S is called *multiplicity free* if its commutant $\pi(S)'$ in $B(\mathcal{H})$ is abelian. Even though a more general formulation of the next result is possible, we emphasize on multiplicity free representations, since in the author's opinion the most interesting examples of holomorphic representations of complex Ol'shanskiĭ semigroups are multiplicity free.

Corollary II.31. *Let $(\pi,\mathcal{H})$ be a holomorphic multiplicity free representation of S and α be the absolute value defined by $\alpha(s) = \|\pi(s)\|$. Then there exists a Radon measure μ on the euclidean space $HW_\alpha \subseteq i\mathfrak{t}^*$ and a direct integral of representations*

$$\Big(\int_{HW_\alpha}^\oplus \pi_\lambda \, d\mu(\lambda), \int_{HW_\alpha}^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)\Big)$$

such that $(\pi,\mathcal{H})$ is unitarily equivalent with $\big(\int_{HW_\alpha}^\oplus \pi_\lambda \, d\mu(\lambda), \int_{HW_\alpha}^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)\big)$.

Proof. Recall that the α-bounded holomorphic representations can be modelled by the representations of the CCR C^*-algebra $C_h^*(S,\alpha)$ (cf. Lemma II.6). In view of this, the assertion follows from [Dix82, Th. 8.6.5] and Theorem II.29(i). ■

Continuous traces

In this final subsection we give a sufficient criterion for $C^*_h(S,\alpha)$ to have continuous trace. We will illustrate this criterion in various examples.

Recall that for each holomorphic highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of S all operators $\pi_\lambda(s)$, $s \in S$, are of trace class (cf. [Ne99, Th. XI.6.1]), and so the notion

$$\Theta_\lambda \colon S \to \mathbb{C}, \quad s \mapsto \operatorname{tr} \pi_\lambda(s)$$

makes sense. We call Θ_λ the *character* of $(\pi_\lambda, \mathcal{H}_\lambda)$ and note that Θ_λ is a holomorphic function on S (cf. [Ne99, Prop. XI.6.4]).

Definition II.32. (cf. [Dix82, Sect. 4.5.2]) Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}_+$ the cone of positive elements in $\mathcal{A}$. Let $\mathfrak{p} \subseteq \mathcal{A}_+$ be the subcone of those elements of $x \in \mathcal{A}_+$ for which the mapping $\widehat{\mathcal{A}} \to [0,\infty]$, $[\pi] \mapsto \operatorname{tr} \pi(x)$ is finite and continuous. Then $\mathfrak{p}$ is the positive portion of a two-sided ideal $\mathfrak{m}$ of $\mathcal{A}$. We say that $\mathcal{A}$ has *continuous trace* if $\mathfrak{m}$ is dense in $\mathcal{A}$. ■

Proposition II.33. *Let S be a complex Ol'shanskiĭ semigroup and $\alpha \in \mathcal{A}_h(S)$ be an absolute value on it. Then the following assertions hold:*

(i) *If there exists an open subset $U \subseteq W \cap \mathfrak{t}$ such that for each $X \in U$ the mapping*

$$\varphi_X \colon HW_\alpha \to \mathbb{R}^+, \quad \lambda \mapsto \Theta_\lambda(\operatorname{Exp}(iX))$$

is continuous, then $C^*_h(S,\alpha)$ has continuous trace.

(ii) *If α is generic, then $C^*_h(S,\alpha)$ has continuous trace.*

Proof. (i) Let $V := \operatorname{Ad}(G).U$ and note that V is an open subset of W. Let $\mathfrak{p}$ and $\mathfrak{m}$ as in Definition II.29. Then $\mathfrak{p} \supseteq j(\operatorname{Exp}(iU))$ by Theorem II.24 and so $\mathfrak{p} \supseteq j(\operatorname{Exp}(iV))$. Since every holomorphic representation of S which vanishes on $\operatorname{Exp}(iV)$ has to be constant, it follows that $j(\operatorname{Exp}(iV))$ generates a dense ideal of $C^*_h(S,\alpha)$ (cf. Lemma II.6). In particular $\mathfrak{m}$ is dense as was to be shown.
(ii) In view of (i), this follows from [Kr99a, Lemma IV.9]. ■

Example II.34. (a) Suppose that G is a simply connected hermitian Lie group. Set $\mathfrak{t}_0 = \mathfrak{t} \cap [\mathfrak{k},\mathfrak{k}]$ and note that $\mathfrak{t} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{t}_0$. According to this decomposition, every element $\lambda \in i\mathfrak{t}^*$ can be written as $\lambda = \lambda_z + \lambda_0$ with $\lambda_z \in i\mathfrak{z}(\mathfrak{k})^*$ and $\lambda_0 \in i\mathfrak{t}_0^*$.

Assume that the absolute value α satisfies the following condition:

(CT) For each $\lambda \in HW_\alpha$ which is not isolated there exits an $\varepsilon > 0$ such that $]1-\varepsilon, 1+\varepsilon[\lambda_z + \lambda_0 \subseteq HW(G,\Delta^+)$.

Note that this condition excludes the first reduction points in $HW(\widetilde{G},\Delta^+)$ and (CT) is exactly the condition for α being generic (cf. [Ne99, Ch. X]). Therefore Proposition II.33(ii) applies and shows that $C^*_h(S,\alpha)$ has continuous trace.
(b) We now discuss (a) in the special case of $G := \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ the universal covering group of $\mathrm{Sl}(2,\mathbb{R})$. We use the notation of Example II.27.

Recall that $\mathfrak{t}^*_{\mathbb{C}}$ was identified with $\mathbb{C}$ and within this identification we have $HW(G,\Delta^+) =]-\infty, 0]$. Then for all $u \in \mathbb{R}^+$ one has

$$\Theta_\lambda(\operatorname{Exp}(iuU)) = \begin{cases} \frac{e^{u\lambda}}{1-e^{2u}} & \text{for } \lambda < 0 \\ 1 & \text{for } \lambda = 0 \end{cases} \tag{2.11}$$

(this is a special case of [Kr99a, Lemma IV.9]; see also [La85, Ch. VII,§4, Th. 5]).

Also recall our identifaction for the absolute values $\mathcal{A}_h(S) \to]-\infty, 0]$, $\alpha \to t_\alpha$. Now it follows from (a) and (2.11) that $C^*_h(S,\alpha)$ has continuous trace if and only if $t_\alpha < 0$, i.e., α is generic. ■

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